Int. J. Heat Mass Transfer. Vol. 11, pp. 141–144. Pergamon Press 1968. Printed in Great Britain

ON DIFFUSION IN PLASMA WITH RECOMBINATION

A. V. LUIKOV, T. L. PERELMAN and S. I. ANISIMOV

Heat and Mass Transfer Institute, Minsk, B.S.S.R., U.S.S.R.

(*Received* 29 *March* 1967)

Аннотация—Рассмотрены закономерности диффузионного распада плазмы при рекомбинации для различных случаев симметрии.

NOMENCLATURE

D, ambipolar diffusivity;

α, recombination coefficient;

ε, $= \dfrac{Q\alpha}{l^{\nu-1}D}$, dimensionless parameter determining the relation between diffusion and concentration.

In STUDYING plasma decay forming, for example, in electric charge or motion of meteorites in the atmosphere, the change in concentration of charged particles due to recombination and diffusion should be taken into account. A number of such problems are, for instance, considered in [1–4].

If the density of the charged particles is not very great, the diffusivity may be considered constant (in the absence of a magnetic field); and the problem consists in solving the diffusion equation:

$$\frac{\partial n}{\partial t} = D\frac{1}{r^{\nu}}\frac{\partial}{\partial r}\left(r^{\nu}\frac{\partial n}{\partial r}\right) - \alpha n^2 \tag{1}$$

with the appropriate boundary conditions; the values $\nu = 0, 1, 2$ correspond to flat, cylindrical and spherical symmetry in the density distribution of the charged particles $n(r, t)$. Below some approximate solutions of equation (1) in an infinite region for $\nu = 0$; 2 will be obtained; the case $\nu = 1$ has been earlier discussed in [4].

In order that the change in the particle density be described by equation (1), the initial distribution of the particle density $n(r, 0)$ should possess an appropriate symmetry. However, the solutions for time moments sufficiently far from the initial one are of interest frequently in practice. Such solutions do not depend upon the details of the initial distribution but only upon some characteristics of it, first of all, upon the integrals

$$Q = \int_0^\infty n(r, 0)\, r^{\nu}\, dr$$

$$Ql^2 = \int_0^\infty n(r, 0)\, r^{\nu+2}\, dr$$

which are accordingly proportional to a definite total number of the particles and the square of the size of the region occupied by the particles at the initial time moment. Since it is not the particle density that is most frequently measured experimentally but the quantities Q and l, it is convenient to obtain, instead of the solution of the initial problem, the dependence of such quantities upon time.

Introduce the dimensionless variables into equation (1)

$$\rho = r/l, \qquad \tau = Dt/l^2,$$

$$f(\rho, \tau) = \frac{\alpha l^2}{D} n(r, t), \qquad \varepsilon = \frac{Q\alpha}{l^{\nu-1}D}.$$

The parameter ε characterizes the relative contribution of diffusion and recombination to the change of the particle concentration. Transforming equation (1), we obtain:

$$\frac{\partial f}{\partial \tau} = \frac{1}{\rho^{\nu}} \frac{\partial}{\partial \rho}\left(\rho^{\nu} \frac{\partial f}{\partial \rho}\right) - f^2; \qquad (2)$$

$$f(\infty, t) = 0; \qquad f(0, t) < \infty;$$

$$\int_0^{\infty} f(\rho, 0)\, \rho^{\nu}\, \mathrm{d}\rho = \int_0^{\infty} f(\rho, 0)\, \rho^{\nu+2}\, \mathrm{d}\rho = \varepsilon. \qquad (3)$$

Multiplying equation (2) by ρ^n and integrating it by terms, we pass to the system of the equations for the moments of the function $f(\rho, \tau)$. If we designate

$$N_{2K}(\tau) = \int_0^{\infty} f(\rho, \tau)\, \rho^{\nu+2K}\, \mathrm{d}\rho,$$

$$M_{2K}(\tau) = \int_0^{\infty} f^2(\rho, \tau)\, \rho^{\nu+2K}\, \mathrm{d}\rho,$$

then the system is written as:

$$\frac{\mathrm{d} N_{2K}}{\mathrm{d}\tau} = 2K(2K + \nu - 1) N_{2K-2} - M_{2K}\,(K = 0, 1 \ldots). \qquad (4)$$

Note that the equation for the $2K$th moment does not contain high-order moments. Therefore, if we are interested in the dependence of only the quantities N_0 and N_2 upon time, we may confine ourselves to the solution of the first two equations of system (4). The function $f(\rho, \tau)$ will be sought in the form:

$$f(\rho, \tau) = g(\tau) \exp\left[-\rho^2 h(\tau)\right], \qquad (5)$$

where $g(\tau)$ and $h(\tau)$ are the unknown functions, chosen so that they should satisfy the first two equations of system (4). For spherical symmetry, the choice of expression (5) for the function $f(\rho, \tau)$ is justified by the fact that, as will be shown below, at $\tau \to \infty$ the particle density distribution in space tends to the diffusion one. For flat symmetry, recombination appears noticeably to influence the particle density distribution at $\tau \to \infty$; nevertheless, as will be seen from a comparison with the similar solution, expression (5) also in this case leads to very exact values of the integrals N_0 and N_2, which is all that is needed in this case.

Substituting system (4) for expression (5) at $K = 0, 1$ gives the system of two ordinary differential equations which is convenient to write down as:

$$\left.\begin{aligned} &\varphi' - \left(\frac{h}{2}\right)^{\frac{\nu+1}{2}} = 0, \\ &\varphi' + 2\varphi\left(4h + \frac{h'}{h}\right) = 0. \end{aligned}\right\} \qquad (6)$$

Here

$$\varphi(\tau) = 1/N_0(\tau)\sqrt{(\pi/2^{\nu+2})}.$$

The initial conditions for system (6) are as follows:

$$\varphi(0) = \frac{1}{\varepsilon}\sqrt{\left(\frac{\pi}{2^{\nu+2}}\right)}; \qquad h(0) = \frac{\nu+1}{2}.$$

Upon integrating the system of equations (6) we have

$$\tau = \int_{\varphi(0)}^{\varphi} \frac{x^{\frac{\nu+1}{4}}\, \mathrm{d}x}{\left(C - 16\frac{\nu-1}{\nu+3} x^{\frac{\nu+3}{4}}\right)^{\frac{\nu+1}{\nu-1}}} \qquad (7)$$

where

$$C = \pi^{\frac{\nu-1}{8}} \varepsilon^{\frac{1-\nu}{4}} 2^{\frac{(1-\nu)(10+\nu)}{8}} \left[(1+\nu)^{\frac{\nu-1}{2}} + \frac{\sqrt{\pi}}{3}\frac{\nu-1}{\nu+3} 2^{\frac{\nu}{2}+2}\right]. \qquad (8)$$

Further, it is convenient to consider the cases of flat and spherical symmetry separately. For flat symmetry, calculating integral (7) we find

$$\tau = F(\varphi) - F(\varphi_0)$$

$$F(x) = \tfrac{4}{5} c x^{\frac{5}{4}} + \tfrac{8}{3} x^2$$

$$C = \pi^{-\frac{1}{8}} \varepsilon^{\frac{1}{4}} 2^{\frac{5}{4}} \left(1 - \frac{4\pi}{3\varepsilon}\right). \qquad (9)$$

Hence it follows that at $\tau \to \infty$

$$N_0(\tau) \simeq \sqrt{\left(\frac{2\pi}{3\tau}\right)}; \qquad N_2(\tau) \simeq \frac{8}{3}\sqrt{\left(\frac{2\pi\tau}{3}\right)}. \qquad (10)$$

Thus, the total number of particles (per unit surface) tends to zero as $1/\sqrt{\tau}$, and the size of the region occupied by the particles [equal to $\sqrt{(N_2/N_0)}$] increases proportionally to $\sqrt{\tau}$. It is interesting to note that in expression (9) the whole information about the initial particle distribution is contained in the constants C and $\varphi(0)$, so that at $\tau \to \infty$ $N_0(\tau)$ does not depend upon the initial number of particles $N(0)$ and the initial size of the distribution [3]. This peculiarity results from the "recombination" nature of the process at $\tau \to \infty$. In order to study it in more detail, note that equation (2) at $\nu = 0$ permits the similar solution of the form:

$$f(\rho, \tau) = \frac{v(\xi)}{\tau + \tau_0}; \qquad \xi = \frac{\rho^2}{\tau + \tau_0}, \tag{11}$$

(formally such solutions are also possible at other values of ν, but only in the case of flat symmetry they possess necessary physical properties). Upon transforming equation (2) we obtain the ordinary differential equation for the function $v(\xi)$

$$\xi v'' + (\tfrac{1}{2} + \xi)\, v' + v - v^2 = 0. \tag{12}$$

The required solution should decrease sufficiently quickly at $\xi \to \infty$ and satisfy the condition $v'(0) = 0$. It is possible to show [3] that the only value of $v(0) = 0{\cdot}690$ corresponds to such a solution. In [3] there is a table of the function $v(\xi)$. Solution (11) depends upon the initial conditions only in terms of the constant τ_0 which (assuming that the initial particle distribution does not differ too much from the similar one*) is equal to

$$\tau_0 = \frac{2{\cdot}105}{\varepsilon^2}.$$

At $\tau \gg \tau_0$ the solution does not depend upon the properties of the initial particle distribution. The total number of particles $v(\xi)$ calculated by means of the function $N_0(\tau)$ at $\tau \to \infty$ appears to be equal to:

$$N_0(\tau) \simeq \frac{1{\cdot}451}{\sqrt{\tau}}.$$

In this expression the coefficient differs from that in equation (10) only by 0·3 per cent.

For spherical symmetry, the integral in equation (7) is also easily calculated; however, the result appears to be cumbersome, and we present only the dependence $oN_0(\tau)$ for large τ:

$$N_0(\tau) \simeq \frac{N_0(\infty)}{1 - \dfrac{N_0(\infty)}{2\sqrt{(2\pi\tau)}}}, \tag{13}$$

where

$$N_0(\infty) = N_0(0)[1 + (5\sqrt{3})/(8\sqrt{\pi})\,\varepsilon]^{-\frac{4}{5}}). \tag{14}$$

From equations (13) and (14) it follows that in the spherical case there occurs "hardening" of ionization, i.e. only the amount of the charged particles available at the initial moment recombines, which is the greater, the larger is the dimensionless recombination coefficient ε. From equation (14) it follows that $N_0(\infty)$ depends only upon $N_0(0)$; this is valid only until the initial number of particles $N_0(0)$ is not very great. At $N_0(0) \to \infty$ the number of the remainder particles depends upon the initial distribution function, and the system of the equations for the first two moments appears to be an insufficient approximation. However, at any $N_0(0)$ in the limit $\tau \to \infty$ the recombination asymptotically "switches itself off", and the particle density distribution corresponds to the solution of a linear diffusion problem normed with $N_0(\infty)$:

$$f(\rho, \tau) \simeq \frac{N_0(\infty)}{2(\sqrt{\pi})\,\tau^{\frac{1}{2}}} \exp(-\rho^2/4\tau). \tag{15}$$

It is also possible to calculate a correction for equation (15) caused by recombination which,

---

* Asymptotically any initial particle distribution converts into the similar one $v(\xi)$; time of approach of a similar regime is determined by nature of the initial particle distribution: if $Ql \gg D/\alpha$, then time of establishment $\tau' \sim l^2/\alpha D$; if, vice versa, $Ql \ll D/\alpha$, then $\tau'' \sim l^2/D$.

as it is easy to see, is of the form $u(\xi)/\tau^2$, and $u(\xi)$ is the solution of the equation

$$\xi u'' + (\tfrac{3}{2} + \xi)\,u' + 2u = \left(\frac{N_0(\infty)}{2\sqrt{\pi}}\right)^2 \exp(-2\xi).$$

The integration of this equation gives

$$u(\xi) = \frac{1}{2}\left(\frac{N_0(\infty)}{2\sqrt{\pi}}\right)^2 \left\{\exp(-2\xi) + \left(\sqrt{\frac{\pi}{\xi}}\right)(\xi - \tfrac{1}{2})\exp(-\xi)\,\mathrm{erf}\sqrt{\xi}\right\}. \quad (16)$$

If $N_0(\tau)$ is now calculated by means of equations (15) and (16), then we have

$$N_0(\tau) \simeq N_0(\infty)\left[1 + \frac{N_0(\infty)}{2\sqrt{(2\pi\tau)}}\right]$$

which at $\tau \to \infty$ coincides with equation (13).

REFERENCES

1. G. A. GRINBERG, On diffusion plasma decay in a magnetic field with non-linear effects, *Zh. Tekh. Fiz.* **33** (12), 1430–1443 (1963).
2. S. I. ANISIMOV and T. L. PERELMAN, Diffusion of charged particles with recombination, Report at the First All-Union Heat and Mass Transfer Conference, Minsk (1961).
3. T. L. PERELMAN and S. I. ANISIMOV, Similar solutions to the problem on recombination plasma decay, *Dokl. Akad. Nauk. SSSR* **172** (4), 817–819 (1967).
4. T. L. PERELMAN and S. I. ANISIMOV, On density distribution of charged particles in meteorite traces, *Dokl. Akad. Nauk. SSSR* **136** (4), 810–812 (1961).

Abstract—The laws of diffusion plasma decay with recombination for different cases of symmetry are considered.

Résumé—On considère les lois de décroissance de la diffusion dans un plasma avec recombinaison pour différents cas de symétrie.

Zusammenfassung—Es werden die Gesetze des Plasmazerfalls bei Diffusion mit Rekombination für verschiedene Symmetriefälle behandelt.